DOUBLE LIE ALGEBROIDS AND THE DOUBLE OF A LIE BIALGEBROID *

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In this paper we define an abstract notion of double Lie algebroid, and consider three major classes of examples.

Firstly, we verify that the double Lie algebroid of a double Lie groupoid, and more generally an $\mathcal{LA}$-groupoid, as constructed in [9], is an abstract double Lie algebroid. A large part of the necessary work for this was done in [10].

Secondly we consider Lie bialgebroids. We show that the double cotangent of a Lie bialgebroid is a double Lie algebroid and, further, that the double cotangent of an *a priori* unrelated pair of Lie algebroid structures on a vector bundle and its dual form a Lie bialgebroid if and only if the double cotangent is a double Lie algebroid. We argue that this is an appropriate form of the Manin triple result for Lie bialgebroids.

Thirdly we consider vacant double Lie algebroids and show that they are equivalent to a matched pair structure on the two side Lie algebroids.

The paper begins with a preliminary study of the triple structures associated with the tangent and cotangent of a double vector bundle.

Several of the results of this paper were announced without proof in [11], which should be read as an introduction to this paper. I am very grateful to Johannes Huebschmann, Yvette Kosmann–Schwarzbach, Alan Weinstein and Ping Xu for conversations on this material at various stages.

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1 COTANGENT TRIPLE VECTOR BUNDLES

A *double vector bundle* is a diagram as in Figure 1(a), in which each side has a vector bundle structure, and the two structures on $\mathcal{A}$ *commute* in the sense that the maps defining each

$$\begin{tikzcd} \mathcal{A} \arrow[r, "\widetilde{q}_H"] \arrow[d, "\widetilde{q}_V"'] & A^V \arrow[d, "q_V"] \\ A^H \arrow[r, "q_H"'] & M \end{tikzcd} \qquad\qquad \begin{tikzcd} \mathcal{A}^{*V} \arrow[r, "\widetilde{q}_V^{(*)}"] \arrow[d, "\widetilde{q}_{*V}"'] & K^* \arrow[d, "q_{K^*}"] \\ A^H \arrow[r, "q_H"'] & M \end{tikzcd}$$

(a) Figure 1: (b)

structure on $\mathcal{A}$ (the bundle projection, zero section, addition and scalar multiplication) are morphisms with respect to the other. This is precisely what is needed to ensure that when four elements $\xi_1, \ldots, \xi_4$ of $\mathcal{A}$ are such that the LHS of

$$(\xi_1 \underset{H}{+} \xi_2) \underset{V}{+} (\xi_3 \underset{H}{+} \xi_4) = (\xi_1 \underset{V}{+} \xi_3) \underset{H}{+} (\xi_2 \underset{H}{+} \xi_4)$$

is defined, then the RHS is also, and they are equal. See [16] or [8, §1].

The *core* of the double vector bundle $\mathcal{A}$ is the intersection K of the kernels of the two projections $\widetilde{q}_H$ and $\widetilde{q}_V$; the vector bundle structures on $\mathcal{A}$ induce a common vector bundle structure on K with base M. The dual $\mathcal{A}^{*V}$ of the vertical bundle structure on $\mathcal{A}$ has, in addition to its standard structure on base A^H, a vector bundle structure on base K^*. The projection is defined by

$$\langle \widetilde{q}_V^{(*)}(\Phi), k\rangle = \langle \Phi, \widetilde{0}_X^V \underset{H}{+} \overline{k}\rangle \tag{1}$$

where $\Phi\colon \widetilde{q}_V^{-1}(X) \to \mathbb{R},\ X \in A_m^H$, and $k \in K_m$. The addition $\underset{H}{+}$ in $\mathcal{A}^{*V} \to K^*$ is defined by

$$\langle \Phi \underset{H}{+} \Phi', \xi \underset{H}{+} \xi'\rangle = \langle \Phi, \xi\rangle + \langle \Phi', \xi'\rangle \tag{2}$$

and the zero above $\kappa \in K_m^*$ is $\widetilde{0}_\kappa^{(*V)}$ defined by

$$\langle \widetilde{0}_\kappa^{(*V)}, \widetilde{0}_x^H \underset{V}{+} \overline{k}\rangle = \langle \kappa, k\rangle$$

where $x \in A_m^V, k \in K_m$. The scalar multiplication is defined in a similar way. These two structures make $\mathcal{A}^{*V}$ a double vector bundle as in Figure 1(b), the *vertical dual of* $\mathcal{A}$. Its core is $(A^V)^*$: the core element $\overline{\psi}$ corresponding to $\psi \in (A_m^V)^*$ is

$$\langle \overline{\psi}, \widetilde{0}_x^H \underset{V}{+} \overline{k}\rangle = \langle \psi, x\rangle.$$

See [17] or [10, §3].

There is also a *horizontal dual* $\mathcal{A}^{*H}$ with sides A^V and K^* and core $(A^H)^*$. The two duals are themselves dual, the pairing being given by

$$\langle \Phi, \Psi\rangle = \langle \Psi, \xi\rangle - \langle \Phi, \xi\rangle \tag{3}$$

where $\Phi \in \mathcal{A}^{*V}$, $\Psi \in \mathcal{A}^{*H}$ have $\widetilde{q}_V^{(*)}(\Phi) = \widetilde{q}_H^{(*)}(\Psi)$ and ξ is any element of $\mathcal{A}$ with $\widetilde{q}_V(\xi) = \widetilde{q}_{*V}(\Phi)$ and $\widetilde{q}_H(\xi) = \widetilde{q}_{*H}(\Psi)$. See [10, §3]; this pairing has also been found in [3].

This pairing could equally well be replaced by its negative. We regard the choice of sign as an extra structure on $\mathcal{A}$ and write $(\mathcal{A}; A^H, A^V; M)^+$ or $(\mathcal{A}; A^V, A^H; M)^-$ for the above convention and $(\mathcal{A}; A^H, A^V; M)^-$ or $(\mathcal{A}; A^V, A^H; M)^+$ for the opposite.

For an ordinary vector bundle (A, q, M) there is the tangent double vector bundle of Figure 2(a); for more detail on this see [1] or [12, §5]. Its vertical dual is the cotangent

$$\begin{tikzcd} TA \arrow[r, "T(q)"] \arrow[d, "p_A"'] & TM \arrow[d, "p"] \\ A \arrow[r, "q"'] & M \end{tikzcd} \qquad\qquad \begin{tikzcd} T^*A \arrow[r, "r"] \arrow[d, "c_A"'] & A^* \arrow[d, "q_*"] \\ A \arrow[r, "q"'] & M \end{tikzcd}$$

(a) (b)

Figure 2:

double vector bundle of Figure 2(b). Its horizontal dual we denote $(T^\bullet A; A^*, TM; M)$; this is canonically isomorphic to $(T(A^*); A^*, TM; M)$ under an isomorphism $I\colon T(A^*) \to T^\bullet A$ given by

$$\langle I(\mathcal{X}), \xi\rangle = \langle\langle \mathcal{X}, \xi\rangle\rangle$$

where $\mathcal{X} \in T(A^*), \xi \in TA$ have $T(q_*)(\mathcal{X}) = T(q)(\xi)$, and $\langle\langle\ ,\ \rangle\rangle$ is the tangent pairing of $T(A^*)$ and TA over TM. See [12, §5].

The structure of $\mathcal{A} = TA$ thus induces a pairing of T^*A and $T(A^*)$ over A^* given by

$$\langle \Phi, \mathcal{X}\rangle = \langle\langle \mathcal{X}, \xi\rangle\rangle - \langle \Phi, \xi\rangle$$

where $\Phi \in T^*A, \mathcal{X} \in T(A^*)$ have $r(\Phi) = p_{A^*}(\mathcal{X})$ and $\xi \in TA$ is chosen so that $T(q)(\xi) = T(q_*)(\mathcal{X})$ and $p_A(\xi) = c_A(\Phi)$. This pairing is nondegenerate by a general result [10, 3.1] so it defines an isomorphism of double vector bundles $R\colon T^*A^* \to T^*A$ by the condition

$$\langle R(\mathcal{F}), \mathcal{X}\rangle = \langle \mathcal{F}, \mathcal{X}\rangle$$

where the pairing on the RHS is the standard one of $T^*(A^*)$ and $T(A^*)$ over A^*. This R preserves the side bundles A and A^* but induces $-\mathrm{id}\colon T^*M \to T^*M$ as the map of cores. In summary we now have the very useful equation

$$\langle \mathcal{F}, \mathcal{X}\rangle + \langle R(\mathcal{F}), \xi\rangle = \langle\langle \mathcal{X}, \xi\rangle\rangle, \tag{4}$$

for $\mathcal{F} \in T^*A^*, \mathcal{X} \in T(A^*), \xi \in TA$, where the pairings are over A^*, A and TM respectively. See [12, 5.5].

Now return to the general double vector bundle in Figure 1(a), denoting the core by K. Since $\mathcal{A}$ has two vector bundle structures, it has two double cotangent bundles. These fit together into a triple structure as the left and rear faces of Figure 3(a), the top face being essentially the cotangent double of the two duals of $\mathcal{A}$. This is, in a sense we will make precise elsewhere, the vertical dual of the tangent prolongation of $\mathcal{A}$ in Figure 3(b). Five of the six

$$
\begin{tikzcd}
T^*\mathcal{A} \arrow[rr] \arrow[d] \arrow[ddr] & & \mathcal{A}^{*H} \arrow[d] \arrow[ddr] & \\
\mathcal{A} \arrow[rr] \arrow[ddr] & & A^V \arrow[ddr] & \\
& \mathcal{A}^{*V} \arrow[rr] \arrow[d] & & K^* \arrow[d] \\
& A^H \arrow[rr] & & M
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
T\mathcal{A} \arrow[rr] \arrow[d] \arrow[ddr] & & T(A^V) \arrow[d] \arrow[ddr] & \\
\mathcal{A} \arrow[rr] \arrow[ddr] & & A^V \arrow[ddr] & \\
& T(A^H) \arrow[rr] \arrow[d] & & TM \arrow[d] \\
& A^H \arrow[rr] & & M
\end{tikzcd}
$$

(a) (b)

Figure 3:

faces in (a) are double vector bundles of types considered already; it is only necessary to verify that the top face is a double vector bundle. (In all diagrams of this type, we take the oblique arrows to be coming out of the page.) The cores of the five faces are known, and we take the core of the top face to be T^*K, in accordance with [10, 1.5]. Taking these cores in pairs, with the edges parallel to them, then gives three double vector bundles: the left–right cores form $(T^*(A^H); A^H, (A^H)^*; M)$, the back–front cores form $(T^*(A^V); A^V, (A^V)^*; M)$ and the top–down cores form $(T^*K; K, K^*; M)$. Each of these *core double vector bundles* has core T^*M.

We will also need to consider the cotangent triples of the two duals of $\mathcal{A}$. Figure 4(a) is the cotangent triple of the double vector bundle $(\mathcal{A}^{*V}; A^H, K^*; M)$; the † denotes the dual over K^*. We use the isomorphisms of double vector bundles

$$
Z_V\colon (\mathcal{A}^{*H})^\dagger \to \mathcal{A}^{*V}, \qquad Z_H\colon (\mathcal{A}^{*V})^\dagger \to \mathcal{A}^{*H}
$$

induced by the pairing (3). Note that Z_V preserves both sides, A^H and K^*, but induces $-\mathrm{id}\colon (A^V)^* \to (A^V)^*$ on the cores, while Z_H preserves K^* and the core $(A^H)^*$, but induces $-\mathrm{id}$ on the sides A^V; this reflects the fact that $Z_V = Z_H^\dagger$, the dual over K^*; see [10, 3.6].

In the case of Figure 2(a), we have $Z_V = R \circ I^\dagger$, where the † dual here is over A^*.

2 ABSTRACT DOUBLE LIE ALGEBROIDS

We come now to the definition of a double Lie algebroid. It will be useful to have a name for a very special case.

Definition 2.1 *An* $\mathcal{LA}$-vector bundle *is a double vector bundle as in Figure* 1(a) *together with Lie algebroid structures on a pair of parallel sides, such that the structure maps of the other pair of vector bundle structures are Lie algebroid morphisms.*

$$
\begin{tikzcd}
T^*(\mathcal{A}^{*V}) \arrow[rr] \arrow[d] \arrow[ddr] & & (\mathcal{A}^{*V})^\dagger \cong \mathcal{A}^{*H} \arrow[d] \arrow[ddr] & \\
\mathcal{A}^{*V} \arrow[rr] \arrow[ddr] & & K^* \arrow[ddr] & \\
& \mathcal{A} \arrow[rr] \arrow[d] & & A^V \arrow[d] \\
& A^H \arrow[rr] & & M
\end{tikzcd}
\qquad
\begin{tikzcd}
T(\mathcal{A}^{*V}) \arrow[rr] \arrow[d] \arrow[ddr] & & T(K^*) \arrow[d] \arrow[ddr] & \\
\mathcal{A}^{*V} \arrow[rr] \arrow[ddr] & & K^* \arrow[ddr] & \\
& T(A^H) \arrow[rr] \arrow[d] & & TM \arrow[d] \\
& A^H \arrow[rr] & & M
\end{tikzcd}
$$

(a) (b)

Figure 4:

For definiteness, take the Lie algebroid structures to be on $\mathcal{A} \to A^H$ and $A^V \to M$.

In the terminology of [8, §4], an $\mathcal{LA}$-vector bundle is an $\mathcal{LA}$-groupoid in which the groupoid structures are vector bundles (and in which the scalar multiplication also preserves the Lie algebroid structures). The core of an $\mathcal{LA}$-groupoid has a Lie algebroid structure induced from the Lie algebroid structure on $\mathcal{A}$ [8, §5]. Each $k \in \Gamma K$ induces $\overline{k} \in \Gamma_{A^H}\mathcal{A}$ defined by $\overline{k}(X) = k(m) \underset{H}{+} \widetilde{0}^V_X$ for $X \in A^H_m$ and the bracket on ΓK is obtained by $\overline{[k,\ell]} = [\overline{k}, \overline{\ell}]$.

Lemma 2.2 *The anchor and the bracket on the core of an $\mathcal{LA}$-vector bundle are zero.*

Proof. The anchor $\widetilde{a}_V \colon \mathcal{A} \to T(A^H)$ is a morphism of double vector bundles and therefore induces a core map $\partial^H \colon K \to A^H$ which, by [8, §5], is a Lie algebroid morphism. Since A^H is abelian, we have $a_K = a_H \circ \partial^H = 0$.

Horizontal scalar multiplication by $t \neq 0$ defines a morphism $\mathcal{A} \to \mathcal{A}$ over $A^H \to A^H$ and therefore induces a map of sections $t_H \colon \Gamma_{A^H}\mathcal{A} \to \Gamma_{A^H}\mathcal{A}$; the Lie algebroid condition then ensures that $[t_H(\xi), t_H(\eta)] = t_H([\xi,\eta])$ for all $\xi, \eta \in \Gamma_{A^H}\mathcal{A}$. Now for $k \in \Gamma K$, $\overline{tk} = t_H(\overline{k})$ and so $\overline{t[k,\ell]} = [t_H(\xi), t_H(\eta)] = [\overline{tk}, \overline{t\ell}] = \overline{t^2[k,\ell]}$. Therefore the bracket must be zero. ∎

We can also apply the calculus developed in [10, §3] for general $\mathcal{LA}$-groupoids. Consider the Poisson structure on $\mathcal{A}^{*V}$. Since it is linear over A^H, the Poisson anchor $\pi^{\#V} \colon T^*(\mathcal{A}^{*V}) \to T(\mathcal{A}^{*V})$ is a morphism of double vector bundles for the left faces of Figure 4, with the corner map $\mathcal{A} \to T(A^H)$ being $\widetilde{a}_V$ and core map $-\widetilde{a}^*_V \colon T^*(A^H) \to \mathcal{A}^{*V}$.

Now applying [10, 3.14], the Poisson structures on $\mathcal{A}^{*V}$ and K^* make $\mathcal{A}^{*V} \to K^*$ a Poisson groupoid; since the Poisson structure on K^* is zero, this is a Poisson vector bundle in the usual sense. Thus $\pi^{\#V}$ is also a morphism of double vector bundles for the rear faces of Figure 4. Denote the corner map $(\mathcal{A}^{*V})^\dagger \to T(K^*)$ by χ_V; since $\pi^{\#V}$ is skew–symmetric, the core map of the rear faces is $-\chi^*_V \colon T^*K^* \to \mathcal{A}^{*V}$. It then follows by a simple argument (as in [10, 2.3]) that $\pi^{\#V}$ is a morphism of triple vector bundles.

We now turn to the general notion of double Lie algebroid. Again consider a double vector bundle as in Figure 1(a). We now assume that there are Lie algebroid structures on all four

sides. The definition comprises three conditions.

Condition I

With respect to the two vertical Lie algebroids, $\mathcal{A} \to A^H$ and $A^V \to M$, the double vector bundle $\mathcal{A}$ is an $\mathcal{LA}$-vector bundle. Likewise, with respect to the two horizontal Lie algebroids, $\mathcal{A} \to A^V$ and $A^H \to M$, the double vector bundle $\mathcal{A}$ is an $\mathcal{LA}$-vector bundle.

Denote the four anchors by $\widetilde{a}_V\colon \mathcal{A} \to T(A^H)$, $\widetilde{a}_H\colon \mathcal{A} \to T(A^V)$, $a_V\colon A^V \to TM$ and $a_H\colon A^H \to TM$. As usual we denote all four brackets by $[\ ,\ ]$; the notation for elements will make clear which structure we are using.

The anchors thus give morphisms of double vector bundles

$$(\widetilde{a}_V; \mathrm{id}, a_V; \mathrm{id})\colon (\mathcal{A}; A^H, A^V; M) \to (T(A^H); A^H, TM; M),$$
$$(\widetilde{a}_H; a_H, \mathrm{id}; \mathrm{id})\colon (\mathcal{A}; A^H, A^V; M) \to (T(A^V); TM, A^V; M)$$

and so define morphisms of their cores; denote these by $\partial^H\colon K \to A^H$ and $\partial^V\colon K \to A^V$.

Now return to $\pi^{\#V}$ and Figure 4. Since the corner map $\mathcal{A} \to T(A^H)$ is $\widetilde{a}_V$, the corner map $A^V \to TM$ is a_V (or $-a_V$ if Z_H is incorporated) and the core map for the front faces is ∂^H. Likewise, since the core map of the left faces is $-\widetilde{a}_V^*$, the core map of the right faces must be $-(\partial^H)^*\colon (A^H)^* \to K^*$ (whether or not Z_H is incorporated). Lastly, the core map of the top faces is $\pi_V^{\#}\colon T^*((A^V)^*) \to T((A^V)^*)$, the anchor for the Poisson structure on $(A^V)^*$ dual to the given Lie algebroid structure on A^V. (These observations are all special cases of [10, §3].)

Each of the maps of the core double vector bundles induces on $T^*M \to (A^V)^*$ the map $-a_V^*$.

Similarly we can analyze $\pi^{\#H}\colon T^*(\mathcal{A}^{*H}) \to T(\mathcal{A}^{*H})$ as a morphism of triple vector bundles, and obtain $\chi_H\colon (\mathcal{A}^{*H})^\dagger \to T(K^*)$.

For Condition II, note first that it is automatic that $\widetilde{a}_V$ is a morphism of Lie algebroids over A^H and that a_V is a morphism of Lie algebroids over M.

Condition II

The anchors $\widetilde{a}_V$ and a_V form a morphism of Lie algebroids with respect to the horizontal structure on $\mathcal{A}$ and the prolongation to $TA^H \to TM$ of the structure on $A^H \to M$. Likewise, the anchors $\widetilde{a}_H$ and a_H form a morphism of Lie algebroids with respect to the vertical structure on $\mathcal{A}$ and the prolongation to $TA^V \to TM$ of the structure on $A^V \to M$.

By Condition I and the discussion preceding it, the Poisson structure on $\mathcal{A}^{*H} \to K^*$ is linear, and therefore induces a Lie algebroid structure on its dual $(\mathcal{A}^{*H})^\dagger$. We use Z_V to transfer this to $\mathcal{A}^{*V} \to K^*$. Similarly the linear Poisson structure on $\mathcal{A}^{*V} \to K^*$ induces a Lie algebroid structure on $(\mathcal{A}^{*V})^\dagger \to K^*$.

Condition III

With respect to these structures, $(\mathcal{A}^{*V}, (\mathcal{A}^{*V})^\dagger)$ is a Lie bialgebroid. Further, $(\mathcal{A}^{*V}; AH, K^*; M)$ is an $\mathcal{LA}$-vector bundle with respect to the horizontal Lie algebroid structures and $(\mathcal{A}^{*H}; K^*, AV; M)$ is an $\mathcal{LA}$-vector bundle with respect to the vertical structures.

Definition 2.3 *A* double Lie algebroid *is a double vector bundle as in Figure* 1(a) *equipped with Lie algebroid structures on all four sides such that the above conditions* I, II, III *are satisfied.*

The notion of Lie bialgebroid was defined in [12] in terms of the coboundary operators associated to A and to A^*; a more efficient and elegant reformulation was then given in [4]. The definition most useful to us here is quoted below in 3.1. For the moment we only need the following.

Suppose that (E, E^*) is a Lie bialgebroid on base P and denote the anchors by e and e_*. Then we take the Poisson structure on P to be $\pi_P^\# = e_* \circ e^*$; this is the opposite to [12], but the same as [4]. It follows that e is a Poisson map (to the tangent lift structure on TP) and e_* is anti–Poisson.

One expects the core of a double Lie algebroid to have a Lie algebroid structure induced from those on $\mathcal{A}$. However, as 2.2 shows, the straightforward embedding of K in terms of core sections yields only the zero structure (see also [9]). Here we obtain the correct structure in terms of its dual.

The anchor of $(\mathcal{A}^{*V})^\dagger$ is χ_V, the appropriate corner map of the Poisson anchor for $\mathcal{A}^{*V}$. On the other hand, the anchor for $\mathcal{A}^{*V}$ itself is $\chi_H \circ Z_V^{-1}$. So the Poisson anchor for K^* is

$$\pi_{K^*}^\# = \chi_V \circ (Z_V^{-1})^\dagger \circ \chi_H^\dagger.$$

Now $Z_V^\dagger = Z_H$ has side map $-\mathrm{id}\colon A^V \to A^V$ and χ_V has side map a_V. The side map of $\chi_H^\dagger$ is the dual of the core map $-(\partial^V)^*$ of χ_H. Thus the side map of $\pi_{K^*}^\#$ is $a_V \circ \partial^V$. One likewise checks that the core map is $-(\partial^H)^* \circ a_H^*$. This proves the first half of the following result.

Proposition 2.4 *The anchor a_K for the Lie algebroid structure on K induced by the Poisson structure on K^* which arises from the Lie bialgebroid structure on $(\mathcal{A}^{*V}, (\mathcal{A}^{*V})^\dagger)$ is $a_V \circ \partial^V = a_H \circ \partial^H$. The maps $\partial^H\colon K \to A^H$ and $\partial^V\colon K \to A^V$ are Lie algebroid morphisms.*

Proof. Since χ_V is the anchor for $(\mathcal{A}^{*V})^\dagger$, it is anti–Poisson into $T(K^*)$. Regarding χ_V as a morphism of the right faces in Figure 4, its core is $-(\partial^H)^*$, which is therefore anti–Poisson. So ∂^H is a morphism of Lie algebroids. ∎

The most fundamental example motivating 2.3 is of course that of the double Lie algebroid of a double Lie groupoid, as constructed in [8], [9]. Most of what is required to verify that the double Lie algebroid of a double Lie groupoid does satisfy 2.3 has been proved in [10], but we recall the details briefly.

In order to proceed, we need to describe the notion of double Lie groupoid in more detail (see [8] and references given there). A double Lie groupoid consists of a manifold S equipped with two Lie groupoid structures on bases H and V, each of which is a Lie groupoid over a common base M, such that the structure maps (source, target, multiplication, identity, inversion) of each groupoid structure on S are morphisms with respect to the other; see Figure 5(a). One should think of elements of S as squares, the horizontal edges of which come from H, the vertical edges from V, and the corner points from M.

Consider a double Lie groupoid $(S; H, V; M)$ as in Figure 5(a). Applying the Lie functor to the vertical structure $S \Longrightarrow H$ gives a Lie algebroid $A_V S \to H$ which has also a groupoid structure over AV obtained by applying the Lie functor to the structure maps of $S \Longrightarrow V$; this

$$\begin{tikzcd} S \arrow[r, Rightarrow] \arrow[d, Rightarrow] & V \arrow[d, Rightarrow] \\ H \arrow[r, Rightarrow] & M \end{tikzcd} \qquad \begin{tikzcd} A_V S \arrow[r, Rightarrow] \arrow[d] & AV \arrow[d] \\ H \arrow[r, Rightarrow] & M \end{tikzcd} \qquad \begin{tikzcd} A^2 S \arrow[r] \arrow[d] & AV \arrow[d] \\ AH \arrow[r] & M \end{tikzcd}$$

(a) (b) (c)

Figure 5:

is the *vertical $\mathcal{LA}$-groupoid* of S [8, §4], as in Figure 5(b). The Lie algebroid of $A_V S \Longrightarrow AV$ is denoted $A^2 S$; there is a double vector bundle structure $(A^2 S; AH, AV; M)$ obtained by applying A to the vector bundle structure of $A_V S \to H$ [9]; see Figure 5(c). Reversing the order of these operations, one defines first the *horizontal $\mathcal{LA}$-groupoid* $(A_H S; AH, V; M)$ and then takes the Lie algebroid $A_2 S = A(A_H S)$. The canonical involution $J_S : T^2 S \to T^2 S$ then restricts to an isomorphism of double vector bundles $j_S : A^2 S \to A_2 S$ and allows the Lie algebroid structure on $A^2 S \to AV$ to be transported to $A_2 S \to AV$. Thus $A_2 S$ is a double vector bundle equipped with four Lie algebroid structures; in [9] we called this the *double Lie algebroid of S*. The core of both double vector bundles $A_2 S$ and $A^2 S$ is AC, the Lie algebroid of the core groupoid $C \Longrightarrow M$ of S [9, 1.6].

Consider $A_2 S$. The structure maps for the horizontal vector bundle $A_2 S \to AV$ are obtained by applying the Lie functor to the structure maps of $A_H S \to V$ and are therefore Lie algebroid morphisms with respect to the vertical Lie algebroid structures. The corresponding statement is true for the vertical vector bundle $A^2 S \to AH$ and this is transported by j_S to $A_2 S \to AH$. Thus Condition I holds.

Let $\overset{\vee}{a}_V : A_2 S \to TAH$ denote the anchor for the Lie algebroid of $A_H S \Longrightarrow AH$. Then, as with any Lie groupoid, $\overset{\vee}{a}_V = A(\overset{\vee}{\chi}_V)$ where $\overset{\vee}{\chi}_V : A_H S \to AH \times AH$ combines the target and source of $A_H S \Longrightarrow AH$. It is easily checked that $\overset{\vee}{\chi}_V$ is a morphism of $\mathcal{LA}$-groupoids over $\chi_V : V \to M \times M$ and $\mathrm{id}: AH \to AH$, and so it follows, by using the methods of [9, §1], that $\overset{\vee}{a}_V$ is a morphism of Lie algebroids over a_V. Similarly one transports the result for the anchors $A^2 S \to TAV$ and $AH \to TM$. Thus Condition II is satisfied.

Now consider the bialgebroid condition. The vertical dual $\mathcal{A}^{*V}$ is $A^*(A_H S)$ and in order to take the dual of this over $A^* C$ we use the isomorphism $j'^V : A^*(A_H S) \to A(A^*_V S)$ of [10, (20)]. This induces $(\mathcal{A}^{*V})^\dagger \cong A^*(A^*_V S)$.

Now the structure on $\mathcal{A}^{*V}$ itself comes from $(\mathcal{A}^{*H})^\dagger$. We have $\mathcal{A}^{*H} = A^\bullet(A_H S)$ and the isomorphism $I_H : A(A^*_H S) \to A^\bullet(A_H S)$ associated to the $\mathcal{LA}$-groupoid $A_H S$ in [10, §3] (see also (5) below) allows us to identify $(\mathcal{A}^{*H})^\dagger$ with $A^*(A^*_H S)$.

So $(\mathcal{A}^{*V}, (\mathcal{A}^{*V})^\dagger)$ is effectively given by $(A^*(A^*_H S), A^*(A^*_V S))$. Now use the isomorphism $\mathcal{D}_H : A^*(A^*_H S) \to A(A^*_V S)$ of [10, 3.9] and we have $(A(A^*_V S), A^*(A^*_V S))$; this is the Lie bialgebroid of $A^*_V S \Longrightarrow A^* C$ which was proved to be a Poisson groupoid in [10, 2.12]. Notice that we started with $A_2 S$, defined in terms of the horizontal $\mathcal{LA}$-groupoid, and ended with the Lie bialgebroid of the dual of the vertical $\mathcal{LA}$-groupoid.

To make this sketch into a proof, one must ensure that the various isomorphisms preserve the Poisson structures involved. Rather than do this, we prove a more general result.

Consider an $\mathcal{LA}$-groupoid as in Figure 6(a); that is, Ω is both a Lie algebroid over G and a Lie groupoid over A, and each of the groupoid structure maps is a Lie algebroid morphism; further, the map $\Omega \to A \underset{M}{\times} G$ defined by the source and the bundle projection, is a surjective submersion. Applying the Lie functor vertically gives a double vector bundle $\mathcal{A} = A\Omega$ as in

$$\begin{tikzcd} \Omega \arrow[r, "\widetilde{q}"] \arrow[d, shift left] \arrow[d, shift right] & G \arrow[d, shift left] \arrow[d, shift right] \\ A \arrow[r, "q_A"'] & M \end{tikzcd} \qquad \begin{tikzcd} A\Omega \arrow[r, "A(\widetilde{q})"] \arrow[d, "\overset{\vee}{q}"'] & AG \arrow[d, "q_G"] \\ A \arrow[r, "q_A"'] & M \end{tikzcd}$$

(a) (b)

Figure 6:

Figure 6(b), with Lie algebroid structures on the vertical sides. It is shown in [9, §1] that the Lie algebroid structure of $\Omega \to G$ may be prolonged to $A\Omega \to AG$.

That the anchor $\overset{\vee}{a}\colon A\Omega \to TA$ for the Lie algebroid of $\Omega \Longrightarrow A$ is a morphism of Lie algebroids over $a_G\colon AG \to TM$ follows as in the case of A_2S above. The anchor $\mathbf{a}\colon A\Omega \to TAG$ for the prolongation structure is $j_G^{-1} \circ A(\widetilde{a})$ where $j_G\colon TAG \to ATG$ is the canonical isomorphism of [12, 7.1]. Since $\widetilde{a}\colon \Omega \to TG$ is a groupoid morphism over $a\colon A \to TM$, and j_G is an isomorphism of Lie algebroids over TM, it follows that Condition II is satisfied. Condition I is dealt with in the same way.

It was shown in [10, §3] that $\Omega^* \Longrightarrow K^*$, the dual groupoid of Ω, together with the Poisson structures on Ω^* and K^* dual to the Lie algebroid structures on Ω and the core K, is a Poisson groupoid. Thus Condition III will follow from the next result.

Theorem 2.5 *The canonical isomorphism of double vector bundles $R = R^{gpd}\colon A^*\Omega^* \to A^*\Omega = \mathcal{A}^{*V}$ is an isomorphism of Lie bialgebroids*

$$(A^*\Omega^*, \overline{A\Omega^*}) \to (\mathcal{A}^{*V}, (\mathcal{A}^{*V})^\dagger)$$

where $(A\Omega^*, A^*\Omega^*)$ is the Lie bialgebroid of the Poisson groupoid $\Omega^* \Longrightarrow K^*$.

We first recall the map R from [10, 3.8]. Associated with $A\Omega$ oriented as in Figure 6(b) there is the pairing of the vertical and horizontal duals (3), which we write in mnemonic form:

$$\langle A^*\Omega, A^\bullet\Omega\rangle_{K^*} = \langle A^\bullet\Omega, A\Omega\rangle_{AG} - \langle A^*\Omega, A\Omega\rangle_A$$

with the subscripts indicating the bases of the pairings. Using the canonical isomorphism $I\colon A\Omega^* \to A^\bullet\Omega$ induced by the pairing $\langle\!\langle\ ,\ \rangle\!\rangle\colon A\Omega^* \underset{AG}{\times} \mathcal{A}\Omega \to \mathbb{R}$, we obtain, as in [10, (18)],

$$\ddagger\, A^*\Omega, A\Omega^* \ddagger = \langle\!\langle A\Omega^*, A\Omega\rangle\!\rangle - \langle A^*\Omega, A\Omega\rangle_A. \tag{5}$$

Now R is defined by $\ddagger \mathcal{X}, R(\mathcal{F})\ddagger = \langle \mathcal{X}, \mathcal{F}\rangle$, where $\mathcal{X} \in A\Omega^*, \mathcal{F} \in A^*\Omega^*$, and the pairing on the RHS is the standard one over K^*. We finally have

$$\langle \mathcal{X}, \mathcal{F}\rangle_{K^*} + \langle R(\mathcal{F}), \Xi\rangle_A = \langle\!\langle \mathcal{X}, \Xi\rangle\!\rangle \tag{6}$$

for compatible $\Xi \in A\Omega$. Equivalently, the canonical isomorphism $Z_V\colon (A^\bullet\Omega)^\dagger \to A^*\Omega$ is given by

$$Z_V = R \circ I^\dagger. \tag{7}$$

The first part of the following result was stated without proof in [10, §3].

Proposition 2.6 (i) *The map $R\colon A^*\Omega^* \to A^*\Omega$ is anti–Poisson from the Poisson structure dual to the Lie algebroid of $\Omega^* \Longrightarrow K^*$ to the Poisson structure dual to the Lie algebroid of $\Omega \Longrightarrow A$.*

(ii) *The map $I\colon A\Omega^* \to A^\bullet\Omega$ is Poisson from the Poisson structure induced on $A\Omega^*$* [19] *by the Poisson groupoid structure on $\Omega^* \Longrightarrow K^*$, to the Poisson structure dual to the prolonged Lie algebroid structure on $A\Omega \to AG$.*

Proof. It suffices [19] to prove that the graph of R is coisotropic in $A^*\Omega^* \times A^*\Omega$. Let $S = \Omega^* \underset{G}{\times} \Omega$ and write $F\colon S \to \mathbb{R}$ for the pairing. Then F is a groupoid morphism, where S is the pullback groupoid over $K^* \underset{M}{\times} A$, and so, as in [10, 3.7], we can apply the Lie functor and get $\langle\!\langle\ ,\ \rangle\!\rangle = A(F)\colon AS \to \mathbb{R}$. This is linear and so defines a section ν of the dual of AS, which is closed since $A(F)$ is a morphism. So by [13, 4.6], the image of ν is coisotropic.

It remains to show that the image of ν coincides with the graph of R. The image of ν consists of those $(\mathcal{F}, \mathcal{X}) \in A^*_\kappa\Omega \times A^*_Y\Omega$ such that

$$\langle(\mathcal{F}, \Phi\rangle, (\mathcal{X}, \Xi)\rangle = A(F)(\mathcal{X}, Xi)$$

for all $(\mathcal{X}, \Xi) \in AS$ compatible with $(\mathcal{F}, \Phi)$. As in [12, 5.5], this equation expands to (6).

We leave the proof of (ii) to the reader. ■

Proof of Theorem 2.5: We must first show that R is an isomorphism of Lie algebroids $A^*\Omega^* \to \mathcal{A}^{*V}$. Now the Lie algebroid structure on $\mathcal{A}^{*V}$ is induced from $(\mathcal{A}^{*H})^\dagger$ via Z_V. So what we have to show is that $Z_V^{-1} \circ R\colon A^*\Omega^* \to (A^\bullet\Omega)^\dagger$ is an isomorphism of Lie algebroids, and this is equivalent to the dual over K^* being Poisson. This dual is, using (7), $I^{-1}\colon A^\bullet\Omega \to A\Omega^*$, and so the result follows from 2.6(ii) above.

Secondly we must show that $R^\dagger\colon (\mathcal{A}^{*V})^\dagger \to \overline{A\Omega^*}$ is an isomorphism of Lie algebroids over K^*. (Note that the minus sign is in the bundle over K^*.) This is equivalent to showing that the dual $R\colon \overline{A^*\Omega^*} \to A^*\Omega$ is Poisson, and this is 2.6(i) above. ■

In summary, we have proved:

Theorem 2.7 *The double Lie algebroid $(A\Omega; A, AG; M)$ of an $\mathcal{LA}$-groupoid $(\Omega; A, G; M)$, as constructed in* [9, §1]*, is a double Lie algebroid as defined in* 2.3.

In particular, the double Lie algebroids $(A_2S; AH, AV; M)$ and $(A^2S; AH, AV; M)$ of a double Lie groupoid $(S; H, V; M)$, as constructed in [9, §2]*, are double Lie algebroids as defined in* 2.3.

Example 2.8 Let A be any Lie algebroid on M. Then $\Omega = A \times A$ has an $\mathcal{LA}$-groupoid structure over $M \times M$ and A, and the associated double Lie algebroid constructed in [9, §1] is $(TA; A, TM; M)$.

The associated duals are $\mathcal{A}^{*V} = T^*A$ and $\mathcal{A}^{*H} = T^\bullet A$. Using R and I as in [12], these can be identified with T^*A^* and $T(A^*)$, as bundles over A^*. The Lie algebroid structure on T^*A^*

is the cotangent of the dual Poisson structure on A^*. The Lie algebroid structure on $T(A^*)$ is the standard tangent bundle structure. This is the standard Lie bialgebroid (T^*P, TP) for $P = A^*$.

Example 2.9 Taking $A = TM$ in the previous example, we see that T^2M is a double Lie algebroid with associated bialgebroid (T^*T^*M, TT^*M). This is a Lie bialgebroid over T^*M, the induced Poisson structure being the standard symplectic structure.

The double Lie algebroids considered in the next two sections do not necessarily have an underlying $\mathcal{LA}$-groupoid.

3 THE DOUBLE LIE ALGEBROID OF A LIE BIALGEBROID

Here we use the following criterion for a Lie bialgebroid.

Theorem 3.1 [12, 6.2] *Let A be a Lie algebroid on M such that its dual vector bundle A^* also has a Lie algebroid structure. Denote their anchors by a, a_*. Then (A, A^*) is a Lie bialgebroid if and only if*

$$T^*(A^*) \xrightarrow{R} T^*(A) \xrightarrow{\pi_A^\#} TA \tag{8}$$

is a Lie algebroid morphism over a_*, where the domain $T^*(A^*) \to A^*$ is the cotangent Lie algebroid induced by the Poisson structure on A^*, and the target $TA \to TM$ is the tangent prolongation of A.

Consider a Lie algebroid A on M together with a Lie algebroid structure on the dual, not *a priori* related to that on A. The structure on A^* induces a Poisson structure on A, and this gives rise to a cotangent Lie algebroid $T^*A \to A$. Equally, the Lie algebroid structure on A induces a Poisson structure on A^* and this gives rise to a cotangent Lie algebroid $T^*A^* \to A^*$. We transfer this latter structure to $T^*A \to A^*$ via R.

There are now four Lie algebroid structures on the four sides of $\mathcal{A} = T^*A$ as in Figure 2(b).

Theorem 3.2 *Let A be a Lie algebroid on M such that its dual vector bundle A^* also has a Lie algebroid structure. Then (A, A^*) is a Lie bialgebroid if and only if $\mathcal{A} = T^*A$, with the structures just described, is a double Lie algebroid.*

PROOF. Assume that (A, A^*) is a Lie bialgebroid. The vertical structure on $\mathcal{A}$ is the cotangent Lie algebroid structure for the Poisson structure on A. The anchor of this is a morphism of double vector bundles $\pi_A^\#: T^*A \to TA$ over $a_*: A^* \to TM$ and id_A, inducing $-a_*^*: T^*M \to A$ on the cores. Now the horizontal structure has the cotangent Lie algebroid structure for the Poisson structure on A^*, transported via $R = R_A: T^*A^* \to T^*A$. So the condition that $\pi_A^\#$ is a morphism of Lie algebroids over a_* with respect to the horizontal structure is precisely 3.1.

On the other hand, the anchor for the horizontal structure is

$$\pi_{A^*}^\# \circ R^{-1}: T^*A \to T(A^*),$$

and this is a morphism of double vector bundles over $a: A \to TM$ and id_{A^*}, inducing $+a^*$ on the cores. Since $R^{-1} = R_{A^*}$, the condition that this anchor be a morphism with respect to the vertical structure is precisely the dual form of 3.1, to which 3.1 is equivalent by [12, 3.10] or [4].

The vertical dual of $\mathcal{A}$ is the tangent double vector bundle as in Figure 2(a). Being the dual of a Lie algebroid, the vertical structure of this has a Poisson structure; this is the tangent lift of the Poisson structure on A [12, 5.6]. The corresponding Poisson tensor is

$$\pi_{TA}^{\#} = J_A \circ T(\pi_A^{\#}) \circ \theta_A^{-1}$$

where $J_A: T^2A \to T^2A$ is the canonical involution for the manifold A and $\theta_A: T(T^*A) \to T^*(TA)$ is the canonical map α of [18], denoted J' in [12, 5.4].

We must check that this Poisson structure coincides with that induced from the dual of $\mathcal{A}^{*H}$. Note that we have $K^* = TM$ here and to avoid confusion we drop the † notation in this case and denote all duals over TM by $^\bullet$. Duals over A^* will be denoted $^{*A^*}$.

Consider $I \circ R^{*A^*}: (T^*A)^{*A^*} \to T^\bullet A$. This preserves the core A^* and the side A^* but reverses the side TM. Define $W = -I \circ R^{*A^*}$ where the minus is for the bundle over A^*. Then $W^\bullet: TA \to (\mathcal{A}^{*H})^\bullet$ and the reader can check that $Z_V^{-1} = \boldsymbol{-}\, W^\bullet$ where the heavy minus is over TM. See alternatively [10, 3.3].

To prove that $Z_V: (\mathcal{A}^{*V})^\bullet \to TA$ is an isomorphism of Lie algebroids over TM we must show that W is an anti–Poisson map. This may be done directly or by observing that $-W$ is, in terms of the double Lie algebroid $\mathcal{A}' = TA$ of 2.8, the map $(Z'_V)^\dagger = Z'_H$.

So we have $\mathcal{A}^{*V} = TA$ and $(\mathcal{A}^{*V})^\bullet = T^\bullet A$ and Condition III follows from the next result. We use I to replace $T^\bullet A$ by TA^*.

Lemma 3.3 *Given that (A, A^*) is a Lie bialgebroid on M, the tangent prolongation structures make (TA, TA^*) a Lie bialgebroid on TM with respect to the tangent pairing.*

Proof. We use the bialgebroid criterion of 3.1. We must prove that

$$T^*(TA^*) \xrightarrow{\widetilde{R}} T^*(TA) \xrightarrow{\pi_{TA}^{\#}} T^2A \tag{9}$$

is a morphism of Lie algebroids over the anchor of TA^* which, by [12, 5.1], is $J_M \circ T(a_*): TA^* \to T^2M$. Here $\widetilde{R}$ is the canonical map R for $TA \to TM$, transported using $I: TA^* \to T^\bullet A$. The domain of (9) is the cotangent Lie algebroid for the Poisson structure on TA^*, which Poisson structure—again by [12, 5.6]—is both the tangent lift of the Poisson structure on A^* and the dual (via I) of the prolongation Lie algebroid structure on TA. The target of (9) is the iterated tangent prolongation of the Lie algebroid structure of A.

Now $\widetilde{R} = \theta_A \circ T(R_A) \circ \theta_{A^*}^{-1}$ and so

$$\pi_{TA}^{\#} \circ \widetilde{R} = J_A \circ T(\pi_A^{\#} \circ R_A) \circ \theta_{A^*}^{-1}.$$

We know that $\pi_A^{\#} \circ R_A: T^*A^* \to TA$ is a morphism of Lie algebroids over a_*, so $T(\pi_A^{\#} \circ R_A)$ is a morphism of the prolongation structures over $T(a_*)$. We need two further observations.

Firstly, for any Poisson manifold, $\theta_P: T(T^*P) \to T^*(TP)$ is an isomorphism of Lie algebroids over TP from the tangent prolongation of the cotangent Lie algebroid structure on T^*P

to the cotangent Lie algebroid of the tangent Poisson structure [9, 2.13]. We apply this to $P = A^*$.

Secondly, $J_A: T^2A \to T^2A$ is a Lie algebroid automorphism over J_M of the iterated prolongation of the given Lie algebroid structure on A.

Putting these facts together, we have that $\pi^{\#}_{TA} \circ \widetilde{R}$ is a Lie algebroid morphism. ∎

Now conversely suppose that A is a Lie algebroid on M and that A^* has a Lie algebroid structure, not *a priori* related to the structure on A. Consider $\mathcal{A} = T^*A$ with the two cotangent Lie algebroid structures arising from the Poisson structures on A^* and A, and suppose that these structures make $\mathcal{A}$ a double Lie algebroid.

Then in particular the anchor of the horizontal structure must be a Lie algebroid morphism with respect to the other structures, as in Condition II, and this is

$$T^*A \xrightarrow{R_{A^*}} T^*A^* \xrightarrow{\pi^{\#}_{A^*}} TA^*$$

That this be a Lie algebroid morphism over $a: A \to TM$ is precisely the dual form of 3.1.

This completes the proof of Theorem 3.2. ∎

Recall the Manin triple characterization of a Lie bialgebra, as given in [6]: Given a Lie bialgebra $(\mathfrak{g}, \mathfrak{g}^*)$ the vector space direct sum $\mathfrak{d} = \mathfrak{g} \oplus \mathfrak{g}^*$ has a Lie algebra bracket defined in terms of the two coadjoint representations. This bracket is invariant under the pairing $\langle X + \varphi, Y + \psi\rangle = \psi(X) + \varphi(Y)$ and both $\mathfrak{g}$ and $\mathfrak{g}^*$ are coisotropic subalgebras. Conversely, if a Lie algebra $\mathfrak{d}$ is a vector space direct sum $\mathfrak{g} \oplus \mathfrak{h}$, both of which are coisotropic with respect to an invariant pairing of $\mathfrak{d}$ with itself, then $\mathfrak{h} \cong \mathfrak{g}^*$ and $(\mathfrak{g}, \mathfrak{h})$ is a Lie bialgebra, with $\mathfrak{d}$ as the double.

Two aspects of this result concern us here. Firstly, it characterizes the notion of Lie bialgebra in terms of a single Lie algebra structure on $\mathfrak{d}$, the conditions being expressed in terms of the simple notion of pairing. Secondly, the roles of the two Lie algebras $\mathfrak{g}$ and $\mathfrak{g}^*$ are completely symmetric; it is an immediate consequence of the Manin triple result that $(\mathfrak{g}, \mathfrak{g}^*)$ is a Lie bialgebra if and only if $(\mathfrak{g}^*, \mathfrak{g})$ is so.

In considering a corresponding characterization for Lie algebroids, the most important difference to note is that the structure on the double is no longer over the same base as the given Lie algebroids. This is to be expected in view of the results of [8, §2] for the double groupoid case. There it is proved that if a double groupoid $(S; H, V; M)$ has trivial core (that is, the only elements of S to have two touching sides which are identity elements, are those which are identities for both structures), then there is a third groupoid structure on S, over base M, called in [8, p.200] the *diagonal structure* and denoted S_D. With respect to this structure on S, the identity maps from H and V into S are morphisms over M, and S as a manifold is $H \underset{M}{\times} V$. This diagonal structure is, in the case where H and V are dual Poisson groups, precisely the structure of the double group. The existence of the diagonal structure, in the general formulation given in [8, §2], depends crucially on the fact that S has trivial core; that is, that S is vacant. Since the core of the double vector bundle T^*A, for A a vector bundle on M, is T^*M, we expect T^*A to possess a Lie algebroid structure over M only when M is a point.

The role played in the bialgebra case by the Lie algebra structure of $\mathfrak{d} = \mathfrak{g} \oplus \mathfrak{g}^*$ is taken, for Lie bialgebroids, by the two structures on T^*A (whose bases are A and A^*). In place of a characterization in terms of a single Lie algebroid structure on T^*A with base M, Theorem

3.2 gives a characterization in terms of the two Lie algebroid structures on T^*A. The role of the pairing in the bialgebra case is taken in 3.2 by the isomorphism R.

The analysis of Lie bialgebras and Poisson Lie groups is usually given in terms of the coadjoint representations and the dressing transformation actions. This was extended by [6], [14] and [5] to the more general situation of matched pairs of Lie groups and Lie algebras, and in [8] to matched pairs of groupoids. In the next section we consider the corresponding results for Lie algebroids.

4 MATCHED PAIRS AND VACANT DOUBLE LIE ALGEBROIDS

The notion of matched pair of Lie algebras was introduced by Kosmann–Schwarzbach and Magri [5], who called them *extensions bicroisées*, by Lu and Weinstein [6], who called them *double Lie algebras*, and by Majid [14], who introduced the term *matched pair*. (In fact, forms of the concept had been found much earlier; see [20] for references.) A matched pair of Lie algebras may be regarded as a triple $(\mathfrak{d}, \mathfrak{g}, \mathfrak{h})$ where the Lie algebra $\mathfrak{d}$ is the vector space direct sum of its subalgebras $\mathfrak{g}$ and $\mathfrak{h}$; a matched pair is thus the notion of Manin triple with the duality aspect removed. A matched pair can be described in terms of a pair of representations, of $\mathfrak{g}$ on $\mathfrak{h}$ and of $\mathfrak{h}$ on $\mathfrak{g}$, subject to twisted derivation conditions. See the references above or 4.1, 4.2 below.

The corresponding concept of matched pair of Lie groups [6], [14] was extended to groupoids in [8]. In [15], Mokri differentiated the twisted automorphism equations of [8] to obtain conditions on a pair of Lie algebroid representations, of A on B and of B on A, which ensure that the direct sum vector bundle $A \oplus B$ has a Lie algebroid structure with A and B as subalgebroids. We quote the following.

Definition 4.1 [15, 4.2] *Let A and B be Lie algebroids on base M, with anchors a and b, and let $\rho\colon A \to \mathrm{CDO}(B)$ and $\sigma\colon B \to \mathrm{CDO}(A)$ be representations of A on the vector bundle B and of B on the vector bundle A. Then A and B together with ρ and σ form a* matched pair *if the following equations hold for all $X, X_1, X_2 \in \Gamma A$, $Y, Y_1, Y_2 \in \Gamma B$:*

$$\begin{aligned} \rho_X([Y_1, Y_2]) &= [\rho_X(Y_1), Y_2] + [Y_1, \rho_X(Y_2)] + \rho_{\sigma_{Y_2}(X)}(Y_1) - \rho_{\sigma_{Y_1}(X)}(Y_2), \\ \sigma_Y([X_1, X_2]) &= [\sigma_Y(X_1), X_2] + [X_1, \sigma_Y(X_2)] + \sigma_{\rho_{X_2}(Y)}(X_1) - \sigma_{\rho_{X_1}(Y)}(X_2), \\ a(\sigma_Y(X)) - b(\rho_X(Y)) &= [b(Y), a(X)]. \end{aligned}$$

Here $\mathrm{CDO}(E)$, for any vector bundle E, is the vector bundle whose sections are the first or zeroth order differential operators $D\colon \Gamma E \to \Gamma E$ for which there is a vector field X on M with $D(f\mu) = fD(\mu) + X(f)\mu$ for all $f \in C(M)$, $\mu \in \Gamma E$. With anchor $D \mapsto X$ and the usual bracket, $\mathrm{CDO}(E)$ is a Lie algebroid (see [7, III§2]).

Proposition 4.2 [15, 4.3] *Given a matched pair, there is a Lie algebroid structure on the direct sum vector bundle $A \oplus B$, with anchor $c(X \oplus Y) = a(X) + b(Y)$ and bracket*

$$[X_1 \oplus Y_1, X_2 \oplus Y_2] = \{[X_1, X_2] + \sigma_{Y_1}(X_2) - \sigma_{Y_2}(X_1)\} \oplus \{[Y_1, Y_2] + \rho_{X_1}(Y_2) - \rho_{X_2}(Y_1)\}. \quad (10)$$

Conversely, if $A \oplus B$ has a Lie algebroid structure making $A \oplus 0$ and $0 \oplus B$ Lie subalgebroids, then ρ and σ defined by $[X \oplus 0, 0 \oplus Y] = -\sigma_Y(X) \oplus \rho_X(Y)$ form a matched pair.

We now show that matched pairs correspond precisely to double Lie algebroids with zero core. The following definition is a natural sequel to [8, 2.11, 4.10].

Definition 4.3 *A double Lie algebroid* $(\mathcal{A}; A^H, A^V; M)$ *is* vacant *if the combination of the two projections,* $(\widetilde{q}_V, \widetilde{q}_H): \mathcal{A} \to A^H \underset{M}{\times} A^V$ *is a diffeomorphism.*

Consider a vacant double Lie algebroid, which we will write here as $(\mathcal{A}; A, B; M)$. Note that $\mathcal{A} \to A^H$ and $\mathcal{A} \to A^V$ are the pullback bundles $q_A^! B$ and $q_B^! A$. The two duals are $\mathcal{A}^{*H} = A^* \oplus B$ and $\mathcal{A}^{*V} = A \oplus B^*$, as vector bundles over M, and the duality is (see [10, 3.4])

$$\langle X + \psi, \varphi + Y \rangle = \langle \varphi, X \rangle - \langle \psi, Y \rangle. \tag{11}$$

The horizontal bundle projection $\widetilde{q}_A: \mathcal{A} \to B$ is a morphism of Lie algebroids over $q_A: A \to M$ and since it is a fibrewise surjection, it defines an action of B on q_A as in [2, §2]. Namely, each section Y of B induces the pullback section $1 \otimes Y$ of $q_A^! B$ and this induces a vector field $\eta(Y) = \widetilde{b}(1 \otimes Y)$ on A, where $\widetilde{b}: \mathcal{A} \to TA$ is the anchor of the vertical structure. By Conditions I and II, $\eta(Y)$ is linear over the vector field $b(Y)$ on M, in the sense of [13, §1]; that is, $\eta(Y)$ is a vector bundle morphism $A \to TA$ over $b(Y): M \to TM$. It follows that $\eta(Y)$ defines covariant differential operators $\sigma_Y^{(*)}$ on A^* and σ_Y on A by

$$\eta(Y)(\ell_\varphi) = \ell_{\sigma_Y^{(*)}(\varphi)}, \quad \langle \varphi, \sigma_Y(X) \rangle = b(Y)\langle \varphi, X \rangle - \langle \sigma_Y^{(*)}(\varphi), X \rangle \tag{12}$$

where $\varphi \in \Gamma A^*$, $X \in \Gamma A$, and ℓ_φ denotes the function $A \to \mathbb{R}$, $X \mapsto \langle \varphi(q_A X), X \rangle$; see [13, §2]. Since $\widetilde{q}_A$ is a Lie algebroid morphism, it follows that σ is a representation of B on the vector bundle A.

Dually, $\widetilde{q}_B$ is a morphism of Lie algebroids over q_B and for each $X \in \Gamma A$ we obtain a linear vector field $\xi(X) \in \mathcal{X}(B)$ over $a(X)$. We likewise define covariant differential operators $\rho_X^{(*)}$ on B^* and ρ_X on B by

$$\xi(X)(\ell_\psi) = \ell_{\rho_X^{(*)}(\psi)}, \quad \langle \psi, \rho_X(Y) \rangle = a(X)\langle \psi, Y \rangle - \langle \rho_X^{(*)}(\psi), Y \rangle. \tag{13}$$

Again, $\rho_X^{(*)}$ and ρ_X are representations of A.

In fact (see [2, §2]) the two Lie algebroid structures on $\mathcal{A}$ are action Lie algebroids determined by the actions $Y \mapsto \eta(Y)$ and $X \mapsto \xi(X)$. It follows that the dual Poisson structures are semi–direct in a general sense, but we prefer to proceed on an ad hoc basis.

For a general vector bundle, the functions on the dual are generated by the linear functions and the pullbacks from the base manifold. In the case of a pullback bundle such as $q_B^! A$, one can refine this description a little further. Namely, if $\pi_B: q_B^! A^* \to A^*$ is $(\varphi, Y) \mapsto \varphi$, and $\widetilde{q}_{A^*}: q_B^! A^* \to B$ is the bundle projection, then the functions on $q_B^! A^*$ are generated by all

$$\ell_X \circ \pi_B, \qquad \ell_\psi \circ \widetilde{q}_{A^*} \qquad \text{and} \qquad f \circ q_B \circ \widetilde{q}_{A^*}$$

where $X \in \Gamma A$, $\psi \in \Gamma B^*$ and $f \in C(M)$. Now the Poisson structure on $q_B^! A^*$ is characterized by

$$\begin{aligned} \{\ell_{X_1} \circ \pi_B,\ \ell_{X_2} \circ \pi_B\} &= \ell_{[X_1, X_2]} \circ \pi_B, & \{\ell_X \circ \pi_B,\ \ell_\psi \circ \widetilde{q}_{A^*}\} &= \ell_{\rho_X^{(*)}(\psi)} \circ \widetilde{q}_{A^*}, \\ \{\ell_X \circ \pi_B,\ f \circ q_B \circ \widetilde{q}_{A^*}\} &= a(X)(f) \circ q_B \circ \widetilde{q}_{A^*}, & \{F_1 \circ \widetilde{q}_{A^*},\ F_2 \circ \widetilde{q}_{A^*}\} &= 0, \end{aligned} \tag{14}$$

where F_1, F_2 are any smooth functions on B. Similarly,

$$\{\ell_{Y_1} \circ \pi_A,\ \ell_{Y_2} \circ \pi_A\} = \ell_{[Y_1,Y_2]} \circ \pi_A, \qquad \{\ell_Y \circ \pi_A,\ \ell_\varphi \circ \widetilde{q}_{B^*}\} = \ell_{\sigma_Y^{(*)}(\varphi)} \circ \widetilde{q}_{B^*},$$
$$\{\ell_Y \circ \pi_A,\ f \circ q_A \circ \widetilde{q}_{B^*}\} = b(Y)(f) \circ q_A \circ \widetilde{q}_{B^*}, \qquad \{G_1 \circ \widetilde{q}_{B^*},\ G_2 \circ \widetilde{q}_{B^*}\} = 0, \tag{15}$$

Now these Poisson structures induce Lie algebroid structures on the direct sum bundles $A \oplus B^*$ and $A^* \oplus B$ over M. Consider first a section $X \oplus \psi$ of $\Gamma(A \oplus B^*)$. Via the pairing (11), this induces a linear function on $q_B^! A^*$, namely

$$\ell^\dagger_{X\oplus\psi} = \ell_X \circ \pi_B - \ell_\psi \circ \widetilde{q}_{A^*}$$

where $\ell^\dagger$ refers to the pairing (11). By following through the equations (14) and (15) one obtains the following.

Lemma 4.4 *The Lie algebroid structure on $A \oplus B^*$ induced as above has anchor $e(X \oplus \psi) = a(X)$ and bracket*

$$[X_1 \oplus \psi_1, X_2 \oplus \psi_2] = [X_1, X_2] \oplus \{\rho^{(*)}_{X_1}(\psi_2) - \rho^{(*)}_{X_2}(\psi_1)\}. \tag{16}$$

The Lie algebroid structure on $A^* \oplus B$ induced as above has anchor $e_*(\varphi \oplus Y) = -b(Y)$ and bracket

$$[\varphi_1 \oplus Y_1, \varphi_2 \oplus Y_2] = \{\sigma^{(*)}_{Y_2}(\varphi_1) - \sigma^{(*)}_{Y_1}(\varphi_2)\} \oplus [Y_2, Y_1]. \tag{17}$$

Thus $A \oplus B^*$ is the semi–direct product (over the base M, in the sense of [7]) of A with the vector bundle B^* with respect to $\rho^{(*)}$. However $A^* \oplus B$ is the opposite Lie algebroid to the semi–direct product of B with A^*.

We can now apply Condition III to $A \oplus B^*$ and $A^* \oplus B$. For brevity write $E = A \oplus B^*$. Recall [12], [4] that (E, E^*) is a Lie bialgebroid if and only if

$$d^E[\varphi_1 \oplus Y_1, \varphi_2 \oplus Y_2] = [d^E(\varphi_1 \oplus Y_1), \varphi_2 \oplus Y_2] + [\varphi_1 \oplus Y_1, d^E(\varphi_2 \oplus Y_2)] \tag{18}$$

for all $\varphi_1 \oplus Y_1, \varphi_2 \oplus Y_2 \in \Gamma E^*$. It follows [12, 3.4] that for any $f \in C(M),\ X \oplus \psi \in \Gamma E$,

$$L_{d^E f}(X \oplus \psi) + [d^{E^*} f, X \oplus \psi] = 0. \tag{19}$$

It is easy to check that

$$d^E f = d^A f \oplus 0, \qquad d^{E^*} f = 0 \oplus d^B f;$$

note that these imply that the Poisson structure induced on M by the Lie bialgebroid (E, E^*) is zero. Now the Lie derivative in (19) is a standard Lie derivative for E^* and so

$$\begin{aligned}
\langle L_{d^E f}(X \oplus \psi), \varphi \oplus Y\rangle &= e_*(d^E f)\langle X \oplus \psi, \varphi \oplus Y\rangle - \langle X \oplus \psi, [d^E f, \varphi \oplus Y]\rangle \\
&= 0 - \langle X \oplus \psi, \sigma^{(*)}_Y(d^A f) \oplus 0\rangle \\
&= \langle d^A f, \sigma_Y(X)\rangle - b(Y)\langle d^A f, X\rangle \\
&= a(\sigma_Y(X))(f) - b(Y)a(X)(f)
\end{aligned} \tag{20}$$

where we used (12) and (17). Expanding out the bracket term in (19) in a similar way, we obtain the third equation in 4.1.

Now consider the bialgebroid equation (18). We calculate this in the case $\varphi_1 = \varphi_2 = 0$, with arguments $0 \oplus \psi_1,\ X_2 \oplus 0$. With these values we refer to (18) as equation $(18)_0$. First we need the following lemma, which is a straightforward calculation.

Lemma 4.5

$$d^E(\varphi \oplus Y)(X_1 \oplus \psi_1, X_2 \oplus \psi_2) = (d^A\varphi)(X_1, X_2) + \langle \psi_1, \rho_{X_2}(Y)\rangle - \langle \psi_2, \rho_{X_1}(Y)\rangle.$$

The LHS of $(18)_0$ is easily seen to be $\langle \psi_1, \rho_{X_2}[Y_2, Y_1]\rangle$. On the RHS, consider the second term first. Regarding the bracket as a Lie derivative, we have

$$\begin{aligned} &\langle L_{0\oplus Y_1}(d^E(0 \oplus Y_2)), (0 \oplus \psi_1) \wedge (X_2 \oplus 0)\rangle \\ &= \ L_{0\oplus Y_1}\langle d^E(0 \oplus Y_2), (0 \oplus \psi_1) \wedge (X_2 \oplus 0)\rangle - \langle d^E(0 \oplus Y_2), L_{0\oplus Y_1}((0 \oplus \psi_1) \wedge (X_2 \oplus 0))\rangle \end{aligned} \tag{21}$$

Since $e_*(0 \oplus Y_1) = -b(Y_1)$, the first term is $-b(Y_1)\langle \psi_1, \rho_{X_2}(Y_2)\rangle$. For the second term we need the following lemma.

Lemma 4.6 *For any $\varphi \oplus Y \in \Gamma E^*$, $X \oplus \psi \in \Gamma E$, we have $L_{\varphi\oplus Y}(X \oplus \psi) = -\sigma_Y(X) \oplus \overline{\psi}$ where for any $Y' \in B$,*

$$\langle \overline{\psi}, Y'\rangle = -b(Y)\langle \psi, Y'\rangle + \langle \sigma^{(*)}_{Y'}(\varphi), X\rangle + \langle \psi, [Y, Y']\rangle.$$

PROOF. This is a Lie derivative for E^* and applying the same device as in (20) we have, for any $\varphi \oplus Y' \in \Gamma E^*$,

$$\langle \varphi' \oplus Y', L_{\varphi\oplus Y}(X \oplus \psi)\rangle = -\langle \varphi', \sigma_Y(X)\rangle + b(Y)\langle \psi, Y'\rangle - \langle \sigma^{(*)}_{Y'}(\varphi), X\rangle + \langle \psi, [Y', Y]\rangle.$$

Setting $Y' = 0$ and $\varphi' = 0$ in turn gives the result. ■

Now expand out the Lie derivative of the wedge product in (21) and apply Lemma 4.6. One obtains for the second term on the RHS of $(18)_0$

$$-\langle \psi_1, [Y_1, \rho_{X_2}(Y_2)]\rangle + \langle \psi_1, \rho_{\sigma_{Y_1}(X_2)}(Y_2)\rangle.$$

The first term on the RHS of $(18)_0$ is easily obtained from this, and combining with the LHS we have the first equation in 4.1. The second equation is obtained in a similar way from the dual form of (18).

This completes the proof of the first part of the following result. The second part is proved essentially by reversing the steps.

Theorem 4.7 *Let $(\mathcal{A}; A, B; M)$ be a vacant double Lie algebroid. Then the two Lie algebroid structures on $\mathcal{A}$ are action Lie algebroids corresponding to actions which define representations ρ, of A on B, and σ, of B on A, with respect to which A and B form a matched pair.*

Conversely, let A and B be a matched pair of Lie algebroids with respect to ρ and σ. Then the action of A on q_B induced by ρ and the action of B on q_A induced by σ define Lie algebroid structures on $\mathcal{A} = A \underset{M}{\times} B$ with respect to which $(\mathcal{A}; A, B; M)$ is a vacant double Lie algebroid.

In the case of a Lie bialgebra $(\mathfrak{g}, \mathfrak{g}^*)$, the Lie bialgebroid associated to the vacant double Lie algebroid is thus $(\mathfrak{g} \oplus \mathfrak{g}_0, \mathfrak{g}^* \oplus \mathfrak{g}_0^*)$ where the subscripts denote the abelianizations. This is of course consistent with 3.2 in the bialgebra case—which is both a bialgebroid and a matched pair.

One other example which should be mentioned briefly is that of affinoids. An *affinoid* [20] may be regarded as a vacant double Lie groupoid in which both side groupoids H and

V are the graphs of simple foliations defined by surjective submersions $\pi_1: M \to Q_1$ and $\pi_2: M \to Q_2$. The corresponding double Lie algebroid was calculated in [9] to be a pair of conjugate flat partial connections adapted to the two foliations. The bialgebroid in this case is $(T^{\pi_1}M \oplus (T^{\pi_2}M)^*, (T^{\pi_1}M)^* \oplus T^{\pi_2}M)$ with semi–direct structures defined by the connections.

Theorem 4.7 provides a diagrammatic characterization of matched pairs of Lie algebroids, directly comparable to the diagrammatic description of matched pairs of group(oid)s given in [8, §2]. In the groupoid case, the twisted multiplicativity equations are fairly unintuitive, and we believe that the derivation of them directly from the vacant double groupoid axioms has been a significant clarification. In the Lie algebroid case the equations in 4.1 are again not simple and, unlike the groupoid case, are defined in terms of sections rather than elements. Nonetheless the characterization given by 4.7 is purely diagrammatic: recall that the characterization 3.1 of a Lie bialgebroid is formulated entirely in terms of the Poisson tensor and the canonical isomorphism R. Thus we have a definition of matched pair which can be formulated more generally in a category possessing pullbacks and suitable additive structure.

We will show elsewhere that it is possible to obtain the Lie algebroid structure on $A \oplus B$ over M directly from the two structures on $\mathcal{A}$.

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